What is Non-Linear Estimation?

You may have heard the terms 'Linear Estimation' and 'Non-Linear Estimation' used in relation to spatial estimation of a resource variable and perhaps wondered exactly what they mean.

Linear Estimation refers simply to any method where the estimate is a function of a linear sum of values that takes the following form:

$$estimate = w_1x_1 + w_2x_2 + w_3x_3 + \cdots$$

The sample values x in this instance are weighted by the respective weights w to produce an estimate.

In the case of a basic linear kriging algorithm such as Ordinary Kriging (OK), the weights are optimised so as to try and minimise the estimation error. If we were to imagine a non-linear estimator, it might look like this:

$$estimate = ax^2 + bx + c$$

You would hopefully recognise that this is a parabolic function, which is clearly non-linear in shape. So, the reason for the terms 'Linear' and 'Non-Linear' just refers to the geometric shape that the estimator function describes. In addition, the sample values are not taken into account when calculating weights in a linear estimate, whereas they are considered in non-linear estimation.

In the context of a mineral resource, why would we want to use a non-linear geostatistical estimator instead of a linear one? What difference does this make and under what conditions would we wish to apply non-linear methods? Finally, what non-linear geostatistical methods are available to us and what is their applicability? To answer these questions we first need to understand what is called the 'volume-variance effect'.

The Volume-Variance Effect

When we sample an ore body for mineralisation, we generally make use of very small samples collected by drilling, face sampling or some other method. You could imagine that the presence or absence of a reasonably sized grain of gold in such a sample would make a large difference to the gold assay grade that it returns.

When we consider a selective mining block that is several metres in size, we are aggregating thousands or even millions of little gold grains, thus smoothing out any highly localised effects and minimising the chance of seeing any extreme grade values. The variability in the sample grades is therefore always going to be much higher than it is for the block grades. This is what is meant by the volume-variance effect – larger volumes will possess a smoother grade distribution and lower grade variance than smaller volumes.

Selective Mining Unit

A Selective Mining Unit (SMU) is the smallest unit of our ore body that we could choose to mine and deliver to a range of destinations, eg. low grade stockpile, high grade stockpile or waste dump. The concept of an SMU is mostly applicable to open pit mining. The optimal shape and size of an SMU block is determined by factors such as the mining method and mining rate, as well as the geometry of the mineralisation.

When in operation, a mine will generally have a grade control sampling programme in place with close spaced sampling that is able to estimate with relatively high accuracy the grade of each SMU. However, it is much more difficult to know what the grade of each SMU will be when we are in the resource definiton stage of a project when we only have wide spaced drilling available. At this point, we can only confidently estimate a block whose width is about half of the drill spacing. Blocks of this size are typically much larger than the optimal SMU size.

If we were to estimate into SMU sized blocks using wide spaced data and a linear estimator such as OK, especially for highly variable minerals like gold, we would tend to produce grade estimates that are far less variable than the actual grades of the SMU's. Since wide spaced drilling is typically used at the feasibility stage, a nasty surprise might await when mining commences and the grade control model shows that the tonnage and grade above cut-off are materially different to what was predicted by the resource model. The mining and processing parameters may therefore turn out to be less than optimal at a point when it would be very difficult or even impossible to make changes, and capital may have been spent on a mining system that is not right-sized for the deposit in question.

Non-linear geostatistics gives us some tools to help overcome the limitations and risks of using linear estimation with wide spaced data points.

Let's look at some commonly used non-linear geostatistical methods.

Uniform Conditioning

Uniform Conditioning (UC) starts with an OK of grade into large blocks, which we call panels (Figure 1).

The panels should be large enough that the linear OK method will produce a result that is consistent with the variance of the panel grades. Remember the larger the block, the more 'smooth' or less variable the grade distribution will be. So, using OK to estimate large panels won't necessarily produce an oversmoothed result, in the way that it would if we tried to estimate the smaller SMU directly.

Once the OK is done, an SMU size, which must be a factor of the larger panel size, is chosen. This SMU size, together with the variogram model based on the sample data, is used to estimate the distribution of SMU grades within each panel. The mean of the SMU grades in each panel corresponds, or is conditioned to, the OK estimate of the panel grade. UC accomplishes all of this using a method called the Discrete Gaussian Model.

The final output of a UC estimate is a set of tonnage, grade and metal values above a set of grade cut-offs which we as the users can choose. These cut-off array variables are stored in the panel block model, meaning that we don't know the location of the SMU grades within the panel, but rather just their distribution.

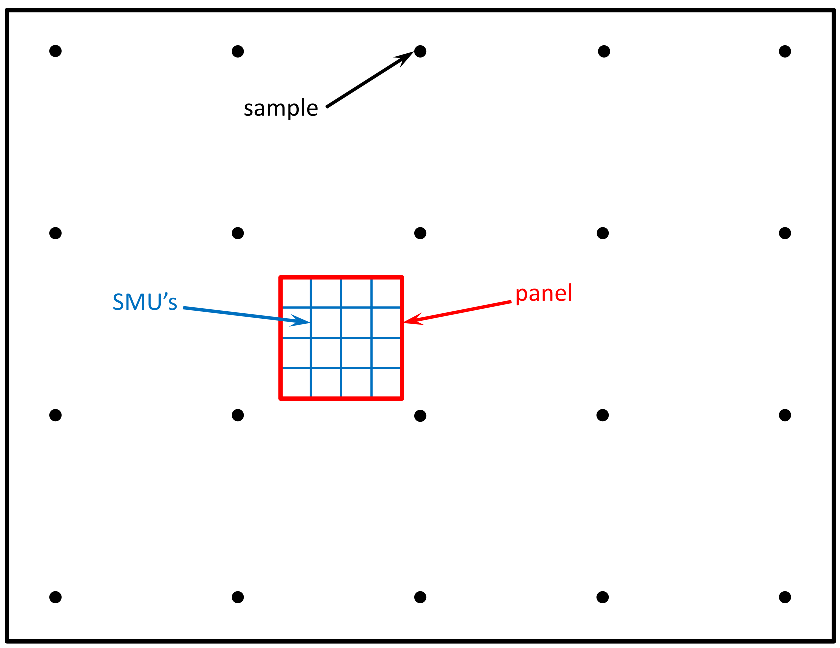

Figure 1: An illustration of panel and SMU block sizes relative to samples and typical relative sample spacing in a resource drilling program.

Multiple Indicator Kriging with Change of Support

Multiple Indicator Kriging (MIK) involves the discretisation of the sample grade distribution using indicator variables. An indicator variable is binary in nature i.e. can only take on a value of 0 or 1, depending on a set of user defined conditions. For instance, we might choose to set the indicator to 1 if the sample has a grade greater than or equal to a given threshold value and to 0 if it is below this threshold.

For MIK, indicator variables are defined on the sample data at increasing grade thresholds to represent the distribution function of the grade variable as a series of discrete steps. These indicators are then each estimated, typically independently using a linear kriging, or sometimes as a linked multivariate system. This estimate is undertaken into panel sized blocks. The result is an estimate of the cumulative distribution function of grade in each panel, but this distribution represents the grade at the sample support size.

In order to proceed to an SMU grade distribution inside each panel, the estimated sample distribution has to be 'squeezed' to lower the variance without altering the mean. There are a number of methods to do this with the simplest being the Affine Correction.

For variables that are believed to have an approximate lognormal distribution (eg. positively skewed variables such as gold grade) the Lognormal or Indirect Lognormal methods can be used. The result is the same set of tonnage, grade and metal above cut-off array variables per panel that a UC produces.

Localisation

Dealing with a set of array variables stored in panels is problematic and cumbersome when undertaking mining studies on a resource block model. Firstly, the granularity of the panels may be insufficient for such work. The array variables also demand fairly complicate block maths to deal with effectively.

In order to overcome these issues, a method was recently developed to use the results of a UC or MIK to estimate individual SMU grades within each panel, resulting in a block model with SMU granularity, but which is still faithful to the results of the non-linear method used to predict the SMU grade distributions. We refer to the outcomes of this post-processing step as Localised Uniform Conditioning (LUC) or Localised Indicator Kriging (LIK).

Conditional Simulation

A range of methods exists to undertake Conditional Simulation (CS) of one or more variables. CS produces a number of different but equiprobable 'realisations' or outcomes. Random variability is introduced during the simulation process. Whilst the result of the simulation at the sample data points returns the exact value of the sample, at any location between samples the results can vary from one realisation to the next, in accordance with the data distribution and the sample variogram model. It is thus possible to generate multiple grade models, which can subsequently be post-processed to obtain useful information, such as the range of uncertainty in an estimate. Importantly, simulation is able to be run on a block size of the user's choice, and if implemented correctly, can deliver a valid representation of the grade-tonnage distribution at the SMU support.

Applicability

Each of the three approaches discussed above are suited to a fairly specific set of conditions and will produce more robust results if matched correctly.

Both UC and CS require that the condition of 'Bi-Gaussianity' is satsified, since both of these techniques rely on the use of the Gaussian or Normal Distribution to achieve their goals.

UC is best suited to the situation where the variable being estimated (grade), is 'diffusive' in character. The condition of diffusion is satisfied when one has to pass through a zone of intermediate grade when moving from an area of low grade to an area of high grade, or vice-versa (Figure 2).

MIK is best applied to a deposit with a 'mosaic' quality to it i.e. grade transitions are abrupt and can occur without the grade boundaries being gradational (Figure 3).

In addition to bi-gaussianity, CS has a further restriction imposed on it in that the mean and the variance of the variable must both be free of a drift or trend across the domain being considered – we refer to this property as Stationarity.

All of these conditions have geostatistical tests which can confirm or deny their existence.

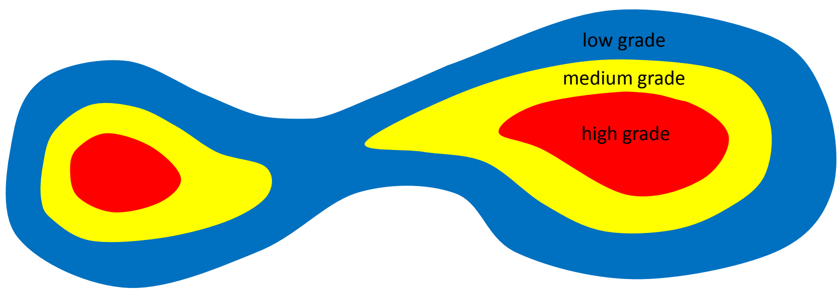

Figure 2: A graphic representation of a diffusive grade model.

Figure 3: A mosaic-like model of mineralisation with mineralised quartz veins (purple) and intervening waste.

Cube's Experience and Capabilities

Cube Consulting has extensive experience and skills in geostatistical estimation and Cube's Geostatisticians frequently apply the non-linear methods described here to generate recoverable resource estimates for mining clients.

Further Reading

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